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 $SL(N + 1, R)$ Toda Solitons in Supergravities

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We construct $(D - 3)$ -brane and instanton solutions using $N \leq 10 - D$ one-form field strengths in D dimensions, and show that the equations of motion can be cast into the form of the $SL(N + 1, R)$ Toda equations. For generic values of the charges, the solutions are non-supersymmetric; however, they reduce to the previously-known multiply-charged supersymmetric solutions when appropriate charges vanish.

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1 Introduction

A convenient procedure for constructing p -brane solitons in string theory or M-theory is first to perform a consistent truncation of the bosonic sector to a subset of the fields that includes the metric, the dilatonic scalars $\vec{\phi}$ and the n -index field strengths F_α that are involved in the solution. In general, we shall concentrate on those theories that are obtained by dimensional reduction of M-theory. The D -dimensional bosonic Lagrangian takes the form

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{2}\sum_{\alpha} e^{-\vec{c}_\alpha\cdot\vec{\phi}} F_\alpha^2 + \mathcal{L}_{FFA} , \quad (1.1)$$

where the “dilaton vectors” \vec{c}_α are constant vectors, characteristic of the dimension D and of the field strengths involved. Their detailed forms can be found in [2].¹ There are also terms, denoted by \mathcal{L}_{FFA} in (1.1), originating from the $F \wedge F \wedge A$ term for the 4-form field strength in $D = 11$. A further complication is that the n -form field strengths F_α are not in general given simply by the exterior derivatives of $(n - 1)$ -form potentials; there are additional Chern-Simons modifications that result from the dimensional reduction process. Thus in general lower-degree potentials also contribute to the n -form field strengths, and conversely, the $(n - 1)$ -form potentials can contribute to fields strengths of degrees higher than n . It is therefore by no means automatic that one can perform a consistent truncation to a sector containing just the metric, dilatonic scalars $\vec{\phi}$, and n -form field strengths F_α that are expressed simply as exterior derivatives of potentials. In particular, it should be emphasised that consistency implies that the vanishing of the truncated fields must be consistent with their own equations of motion, and that it is not sufficient that they simply be absent from the Lagrangian.

There are two reasons why it is nonetheless preferable to try to work only with sets of fields for which the above consistent truncation can be performed. The first is that it is much simpler to solve the equations when the Chern-Simons and FFA terms are not active. The second reason is that in fact many of the solutions where the Chern-Simons and FFA terms play rôle are nothing but U-duality rotations of simpler solutions where the Chern-Simons and FFA terms are not active. Thus a convenient strategy for enumerating solutions is first to look for those where the Chern-Simons and FFA terms do not contribute, and then,

¹We shall follow the notation of [2], in which internal compactified indices are denoted by i, j, \dots , running over $11 - D$ values. Thus in D dimensions there is the 4-form F_4 , and 3-forms $F_3^{(i)}$, 2-forms $F_2^{(ij)}$ and 1-forms $F_1^{(ijk)}$ coming from the 4-form in $D = 11$, and 2-forms $\mathcal{F}_2^{(i)}$ and 1-forms $\mathcal{F}_1^{(ij)}$ coming from the $D = 11$ metric. The 1-forms $\mathcal{F}_1^{(ij)}$, coming from the further dimensional reduction of the 2-forms $\mathcal{F}_2^{(i)}$ in higher dimensions, are defined only for $j > i$.

if desired, to act on these with U-duality in order to fill out entire U-duality multiplets. (For example, the general extremal black hole solutions in $D = 4$ heterotic string and type IIA string have been constructed in [3, 4].) Of course not all of the simpler solutions lie in the same multiplet; for example there can be solutions with different fractions of unbroken supersymmetry, achieved by using different combinations of n -form field strengths, which obviously cannot be related by U duality.

If the dilaton vectors for a set of N field strengths F_α of rank $n \geq 2$ satisfy the dot products

$$M_{\alpha\beta} \equiv \vec{c}_\alpha \cdot \vec{c}_\beta = 4\delta_{\alpha\beta} - \frac{2(n-1)(D-n-1)}{D-2}, \quad (1.2)$$

then either they themselves, or a set related to them by the action of the Weyl group of the U duality group, admit p -brane solutions where the Chern-Simons and FFA terms are not active [2]. The maximum value of N depends on the rank of the field strengths, and on the dimension D . For example, for 2-form field strengths, which can be used to construct black holes or $(D-4)$ -branes, $N_{\max} = 2$ for $6 \leq D \leq 9$; $N_{\max} = 3$ in $D = 5$; and $N_{\max} = 4$ in $3 \leq D \leq 4$. In fact, we can perform a further truncation to the single-scalar Lagrangian

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2n!}e^{-a\phi}F^2 \quad (1.3)$$

where a , ϕ and F are given by [2]

$$\begin{aligned} a^2 &= \left(\sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \right)^{-1}, & \phi &= a \sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \vec{c}_\alpha \cdot \vec{\phi}, \\ F_\alpha^2 &= a^2 \sum_{\beta} (M^{-1})_{\alpha\beta} F^2, \end{aligned} \quad (1.4)$$

The parameter a can be conveniently re-expressed as

$$a^2 = \Delta - \frac{2d\tilde{d}}{D-2}, \quad (1.5)$$

where $\Delta = 4/N$. For elementary solutions $d = n - 1$, while $d = D - n - 1$ for solitonic solutions, with $\tilde{d} \equiv D - d - 2$ in both cases. All these solutions are supersymmetric, preserving 2^{-N} of the supersymmetry for $N \leq 3$, and $1/8$ for $N = 4$. In these single-scalar p -branes, the charges carried by each field strength F_α are equal. They can be generalised to multi-scalar solutions where the N charges become independent parameters [5]. It was observed in [6] that the equations of motion describing these multi-scalar p -branes could be cast into the form of N Liouville equations, with the constraint that the total Hamiltonian vanishes. Note that these simpler solutions, where the Chern-Simons and FFA terms vanish, can be oxidised to $D = 11$ where they can be interpreted as M-branes, intersecting

M-branes [7-12], or boosted intersecting M-branes [12]. It should be remarked that there also exist many other choices of sets of field strengths for which the dilaton vectors do not satisfy the conditions (1.2) [2]. For any such set of field strengths, it seems that there can be no simple solutions where the Chern-Simons or FFA terms can be neglected, and hence the non-supersymmetric solutions described in [2] cannot be embedded in the supergravity theory.

The situation for 1-form field strengths is a little different. Their potentials are 0-forms, subject to constant shift symmetries. They may be scalars or pseudo-scalars. The associated p -branes have either $p = -1$ in the elementary case, or $p = D - 3$ in the solitonic case. The former can be viewed as instantons, and require that the D -dimensional spacetime be Euclideanised to positive-definite signature. There are again consistent truncations possible when the dilaton vectors of the retained field strengths satisfy (1.2). In this case, we have $N_{\max} = 2$ for $7 \leq D \leq 8$; $N_{\max} = 4$ for $5 \leq D \leq 6$; $N_{\max} = 7$ for $D = 4$ and $N_{\max} = 8$ for $D = 3$. All of the solutions preserve partial supersymmetry.

In this paper, we shall show that further consistent truncations are possible for 1-form field strengths, in certain cases where the dilaton vectors do not satisfy (1.2), namely if the dot products of the N field strengths instead satisfy the relation

$$M_{\alpha\beta} = 4\delta_{\alpha\beta} - 2\delta_{\alpha,\beta+1} - 2\delta_{\alpha,\beta-1} . \quad (1.6)$$

This is in fact twice the Cartan matrix for $SL(N+1, R)$. As we shall show, this has the consequence that the equations of motion of the consistently-truncated system can be cast into the form of the $SL(N+1, R)$ Toda equations, with the Hamiltonian constrained to vanish. We are thus able to obtain explicit multi-scalar solutions in these cases. They can also be further reduced to single-scalar solutions in which the N charges occur in fixed ratios, determined by (1.4). These single-scalar solutions have

$$a^2 = \Delta = \frac{24}{N(N+1)(N+2)} . \quad (1.7)$$

It is interesting to note that all the p -brane solutions for which the Chern-Simons and FFA terms are not active seem to be associated with completely-integrable systems of equations. As we mentioned above, the supersymmetric solutions with $\Delta = 4/N$ arise as solutions of diagonalised systems of Liouville equations. In addition, there is a $\Delta = 4$ (*i.e.* $a = \sqrt{3}$) dyonic black hole in $D = 4$ [13, 14], which arises as a solution of the $SL(3, R)$ Toda equations [6]. This, in common with the new $SL(N+1, R)$ Toda solutions we shall obtain below, is non-supersymmetric. All of the non-supersymmetric examples have

negative binding energy, in the sense that they can decay into basic $\Delta = 4$ constituents whose total mass is smaller when widely separated.

2 Toda $(D - 3)$ -branes

Solitonic $(D - 3)$ -branes arise as solutions purely of the scalar and pseudoscalar sector of the supergravity theory. The bosonic Lagrangian for this sector in D dimensions can be written in the form

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{2}\sum_{\alpha}e^{-\vec{c}_{\alpha}\cdot\vec{\phi}}F_{\alpha}^2. \quad (2.1)$$

In this formulation, the spin-0 fields have been divided into two categories, namely dilatonic scalars $\vec{\phi} = (\phi_1, \phi_2, \dots)$ which appear in the exponentials, and the rest, which can be viewed as 0-form potentials for the 1-form field strengths F_{α} . In general the structure of these field strengths is complicated, owing to the Chern-Simons modifications coming from dimensional reduction. In the case of the dimensional reduction of $D = 11$ supergravity, the constant dilaton vectors \vec{c}_{α} that characterise the dilaton couplings, and the Chern-Simons modifications to the field strengths F_{α} , may be found in [2].

The simplest kind of $(D - 3)$ -brane is obtained by setting all except one of the field strengths F_{α} to zero, in which case the Lagrangian (2.1) can be consistently truncated to

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{-a\phi}F^2, \quad (2.2)$$

where $a = 2$, and $F = d\chi$ since in this truncation all the Chern-Simons modifications vanish. The solution for the $(D - 3)$ -brane is given by [15, 16]

$$\begin{aligned} ds^2 &= \eta_{\mu\nu}dx^{\mu}dx^{\nu} + H^{4/a^2}(dr^2 + r^2d\theta^2), \\ e^{a\phi/2} &= H = 1 + k \log r, \quad \chi = 4Q\theta, \end{aligned} \quad (2.3)$$

where $Q = k/(2a)$. The shift symmetry $\chi \rightarrow \chi + 1$ implies that the charge must be quantised, *i.e.* $Q = j/(8\pi)$.

The metric (2.3) is not asymptotically flat. However, the reason for this is simply that, as usual for supersymmetric p -brane solutions, the metric and dilaton are given in terms of an harmonic function H on the transverse space, and in this case the transverse space is two-dimensional, implying a logarithmic harmonic function. In fact these solutions can in general be obtained by the process of vertical dimensional reduction from an asymptotically well-behaved $(D - 3)$ -brane in $(D + 1)$ dimensions [17]. In this process, one constructs a multi-centered p -brane in the higher dimension, with a continuum of centers lying along

the axis of the extra dimension of the three-dimensional transverse space. As with the analogous construction of the potential for a uniform line of charge in electrostatics, the integration over a line of $1/r$ harmonic functions in R^3 gives the $\log r$ harmonic function in R^2 . The pathologies associated with the asymptotically-divergent harmonic function can thus be avoided if the solution is vertically oxidised to the higher dimension. In particular, since the mass per unit spatial p -volume is always preserved under dimensional reduction, one can attach a formal meaning to the mass per unit $(D - 3)$ -volume in D dimensions, despite the absence of an asymptotically Minkowskian spacetime structure. For a metric of the form $ds^2 = dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B}(dr^2 + r^2 d\theta^2)$, the ADM mass per unit $(D - 3)$ -volume is formally given by

$$m = \frac{1}{2} \frac{dB}{d\rho} \Big|_{\rho \rightarrow 0} , \quad (2.4)$$

where $\rho = \log r$. The anticommutator of supercharges \mathcal{Q}_ϵ defines the Bogomol'nyi matrix $\mathcal{M} = \{\mathcal{Q}_{\epsilon_1}, \mathcal{Q}_{\epsilon_2}\} = \epsilon_1^\dagger \mathcal{M} \epsilon_2$, whose zero eigenvalues correspond to unbroken components of supersymmetry. For p -brane solutions using the $\mathcal{F}_1^{(ij)}$ 1-form field strengths, \mathcal{M} is given by [2]

$$\mathcal{M} = m\mathbb{1} + \frac{1}{2} q_{ij} \Gamma_{\hat{1}\hat{2}ij} + \frac{1}{2} p_{ij} \Gamma_{ij} , \quad (2.5)$$

where q_{ij} denote the magnetic charges in the $(D - 3)$ -brane case, with $\hat{1}, \hat{2}$ being the transverse-space indices, and p_{ij} denote the electric charges in the instanton case. For the $(D - 3)$ -brane given in (2.3), just one of the magnetic charges, *e.g.* q_{12} , is non-zero, and the eigenvalues μ of the Bogomol'nyi matrix $\mathcal{M} = m\mathbb{1} + Q\Gamma_{\hat{1}\hat{2}12}$ are given by $\mu = m \pm Q$. Since $m = Q$ for this solution, we see that it saturates the Bogomol'nyi bound, and that it preserves $\frac{1}{2}$ of the supersymmetry.

Further supersymmetric $(D - 3)$ -branes arise under certain circumstances when more than one field strength F_α carries a charge. Specifically, if N field strengths have dilaton vectors \vec{c}_α that satisfy the condition

$$M_{\alpha\beta} \equiv \vec{c}_\alpha \cdot \vec{c}_\beta = 4\delta_{\alpha\beta} , \quad (2.6)$$

then there can be a solution given by [5]

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu + \left(\prod_{\alpha=1}^N H_\alpha \right) (dr^2 + r^2 d\theta^2) , \\ e^{\varphi_\alpha} &= H_\alpha = 1 + k_\alpha \log r , \quad \chi_\alpha = 4Q_\alpha \theta , \end{aligned} \quad (2.7)$$

where $Q_\alpha = \frac{1}{4}k_\alpha$, and $\varphi_\alpha \equiv \vec{c}_\alpha \cdot \vec{\phi}$. To be precise, not every set of field strengths whose dilaton vectors satisfy (1.2) will give a solution of this form, because the Chern-Simons

terms will in general contribute. However, there always exists some choice of field strengths for which the Chern-Simons terms will make no contribution. The full discrete set of choices of field strengths that satisfy (2.6) for a given N forms a multiplet under the Weyl group of the U duality group [18]. The mass per unit $(D - 3)$ volume is equal to the sum of the charges Q_α , and again saturates the Bogomol'nyi bound. The solutions preserve varying amounts of supersymmetry depending upon the number of N of non-vanishing charges. For example, they preserve $\frac{1}{4}$ for $N = 2$, and $\frac{1}{8}$ for $N = 3$. Further details may be found in [2].

Let us now turn to the construction of the $SL(N + 1, R)$ Toda solutions. These make use of the 1-form field strengths $\mathcal{F}_1^{(ij)}$ that come from the dimensional reduction of the $D = 11$ metric. The full Lagrangian can be consistently truncated to a sector involving just these, and the metric and dilaton scalars:

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{2}\sum_{i < j} e^{-\vec{b}_{ij}\cdot\vec{\phi}} (\mathcal{F}_1^{(ij)})^2 . \quad (2.8)$$

The dilaton vectors \vec{b}_{ij} are given in [2], and satisfy

$$\vec{b}_{ij} \cdot \vec{b}_{k\ell} = 2\delta_{ik} + 2\delta_{j\ell} - 2\delta_{i\ell} - 2\delta_{jk} . \quad (2.9)$$

The full expressions for the field strengths, including Chern-Simons corrections, are [2]

$$\mathcal{F}_1^{(ij)} = \gamma^{kj} d\mathcal{A}_0^{(ik)} , \quad (2.10)$$

where

$$\gamma^{ij} \equiv ((1 + \mathcal{A}_0)^{-1})^{ij} = \delta^{ij} - \mathcal{A}_0^{(ij)} + \mathcal{A}_0^{(ik)} \mathcal{A}_0^{(kj)} + \dots . \quad (2.11)$$

Note that the 0-form potentials $\mathcal{A}_0^{(ij)}$, like the 1-form field strengths, exist only for $i < j$, and so the series in (2.11) terminates after a finite number of terms.

It is easy to verify from the above that the field strengths $\mathcal{F}_1^{(i,i+1)}$ have no Chern-Simons corrections. Denoting these by $\mathcal{F}_\alpha \equiv \mathcal{F}_1^{(\alpha,\alpha+1)} = d\chi_\alpha$, and their associated dilaton vectors by $\vec{c}_\alpha \equiv \vec{b}_{\alpha,\alpha+1}$, which satisfy the dot product relations (1.6), we see that Lagrangian (2.8) can be consistently truncated to

$$e^{-1}\mathcal{L} = R - \frac{1}{2} \sum_{\alpha,\beta=1}^N (M^{-1})_{\alpha\beta} \partial_M \varphi_\alpha \partial^M \varphi_\beta - \frac{1}{2} \sum_{\alpha=1}^N e^{-\varphi_\alpha} (\partial\chi_\alpha)^2 , \quad (2.12)$$

where $\varphi_\alpha = \vec{c}_\alpha \cdot \vec{\phi}$. The maximum value of N in D dimensions is clearly given by $N_{\max} = 10 - D$. We proceed by making the standard metric and field strength ansätze

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu + e^{2B(r)} (dr^2 + r^2 d\theta^2) , \\ \chi_\alpha &= 4Q_\alpha \theta . \end{aligned} \quad (2.13)$$

Substituting into the equations of motion following from (2.12), we obtain

$$\varphi''_\alpha = -8 \sum_\beta M_{\alpha\beta} Q_\beta^2 e^{-\varphi_\alpha} , \quad B = \sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \varphi_\alpha , \quad (2.14)$$

$$\sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} \varphi'_\alpha \varphi'_\beta = 16 \sum_\alpha Q_\alpha^2 e^{-\varphi_\alpha} , \quad (2.15)$$

where a prime denotes a derivative with respect to $\rho = \log r$. Making the redefinition $\Phi_\alpha = -2 \sum_\beta (M^{-1})_{\alpha\beta} \varphi_\beta$, these equations become

$$\begin{aligned} \Phi''_\alpha &= 16 Q_\alpha^2 \exp\left(\frac{1}{2} \sum_\beta M_{\alpha\beta} \Phi_\beta\right) , \quad B = -\frac{1}{2} \sum_\alpha \Phi_\alpha , \\ \sum_{\alpha,\beta} M_{\alpha\beta} \Phi'_\alpha \Phi'_\beta &= 64 \sum_\alpha Q_\alpha^2 \exp\left(\frac{1}{2} \sum_\beta M_{\alpha\beta} \Phi_\beta\right) . \end{aligned} \quad (2.16)$$

The further redefinition $\Phi_\alpha = q_\alpha - 4 \sum_\beta (M^{-1})_{\alpha\beta} \log(4Q_\beta)$ removes the charges from the equations, giving

$$\begin{aligned} q''_1 &= e^{2q_1 - q_2} , \\ q''_2 &= e^{-q_1 + 2q_2 - q_3} , \\ q''_3 &= e^{-q_2 + 2q_3 - q_4} , \\ &\dots \\ q''_N &= e^{-q_{N-1} + 2q_N} . \end{aligned} \quad (2.17)$$

These are precisely the $SL(N+1, R)$ Toda equations. The solution is subject to the further constraint (2.15), which, in terms of the q_α , becomes the constraint that the Hamiltonian

$$\mathcal{H} = 4 \sum_{\alpha,\beta} (M^{-1})_{\alpha\beta} p_\alpha p_\beta - \sum_\alpha \exp\left(\frac{1}{2} \sum_\beta M_{\alpha\beta} q_\beta\right) \quad (2.18)$$

for the Toda system (2.17) vanishes.

The general solution to the $SL(N+1, R)$ Toda equations is presented in an elegant form in [19]:

$$e^{-q_\alpha} = \sum_{k_1 < \dots < k_\alpha} f_{k_1} \dots f_{k_\alpha} \Delta^2(k_1, \dots, k_\alpha) e^{(\mu_{k_1} + \dots + \mu_{k_\alpha})\rho} , \quad (2.19)$$

where $\Delta^2(k_1, \dots, k_\alpha) = \prod_{k_i < k_j} (\mu_{k_i} - \mu_{k_j})^2$ is the Vandermonde determinant, and f_k and μ_k are arbitrary constants satisfying

$$\prod_{k=1}^{N+1} f_k = -\Delta^{-2}(1, 2, \dots, N+1) , \quad \sum_{k=1}^{N+1} \mu_k = 0 . \quad (2.20)$$

The Hamiltonian, which is conserved, takes the value $\mathcal{H} = \frac{1}{2} \sum_{k=1}^{N+1} \mu_k^2$.

The solution (2.19) in general involves exponential functions of ρ . Furthermore, the vanishing of the Hamiltonian implies that the parameters μ_k , and hence the solutions, will in general be complex. However, there exists a limit, under which all the μ_k constants vanish, which achieves a vanishing Hamiltonian and real solutions that are finite polynomials in ρ . Since we are constructing $(D - 3)$ -branes in $D \geq 3$, it follows that we are interested in obtaining solutions to the $SL(N + 1, R)$ Toda equations for $N \leq 7$. When $N = 1$, the Toda system reduces to the Liouville equation, giving rise to the usual single field strength solution that preserves 1/2 the supersymmetry, namely

$$e^{-q_1} = 1 + 4Q\rho . \quad (2.21)$$

Note that since there is only a single independent μ parameter when $N = 1$, which has to be zero by the Hamiltonian constraint, (2.21) is in fact the only solution in this case.

For $N = 2$, we find that the polynomial solution to the $SL(3, R)$ Toda equations (2.17) is

$$\begin{aligned} e^{-q_1} &= a_0 + a_1 \rho + \frac{1}{2} \rho^2 , \\ e^{-q_2} &= a_1^2 - a_0 + a_1 \rho + \frac{1}{2} \rho^2 , \end{aligned} \quad (2.22)$$

where a_0 and a_1 are constants that are related to the charge parameters Q_1 and Q_2 , on using the boundary condition that the dilatonic scalars, and hence Φ_α , vanish “asymptotically” (*i.e.* at $\rho = 0$). Thus we have

$$a_0 = \frac{1}{16} Q_1^{-4/3} Q_2^{-2/3} , \quad a_1 = \frac{1}{4} Q_1^{-2/3} Q_2^{-2/3} (Q_1^{2/3} + Q_2^{2/3})^{1/2} , \quad (2.23)$$

which implies that the metric is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + T_1 T_2 (dr^2 + r^2 d\theta^2) , \quad (2.24)$$

where

$$\begin{aligned} T_1 &= 1 + 4Q_1^{2/3} (Q_1^{2/3} + Q_2^{2/3})^{1/2} \rho + 8Q_1^{4/3} Q_2^{2/3} \rho^2 , \\ T_2 &= 1 + 4Q_2^{2/3} (Q_1^{2/3} + Q_2^{2/3})^{1/2} \rho + 8Q_2^{4/3} Q_1^{2/3} \rho^2 , \end{aligned} \quad (2.25)$$

and $\rho = \log r$. It follows from (2.4) that the mass per unit $(D - 3)$ -volume is given by

$$m = (Q_1^{2/3} + Q_2^{2/3})^{3/2} . \quad (2.26)$$

This rather unusual looking mass formula in fact also arises in the $a = \sqrt{3}$ four-dimensional dyonic black hole [14]. In that case also, the equations of motion can be cast into the

form of the $SL(3, R)$ Toda equations [6]. For non-vanishing Hamiltonian the black hole is non-extremal, becoming extremal when the Hamiltonian vanishes. The mass formula (2.26) implies that the solution describes a system with negative binding energy, since the total mass of the widely-separated constituents is given by $m_\infty = Q_1 + Q_2$, which is smaller than m . The Bogomol'nyi matrix in this case is $\mathcal{M} = m\mathbb{1} + Q_1\Gamma_{\hat{1}\hat{2}12} + Q_2\Gamma_{\hat{1}\hat{2}23}$, and therefore its eigenvalues are

$$\mu = m \pm \sqrt{Q_1^2 + Q_2^2}. \quad (2.27)$$

It follows from (2.26) that the μ is strictly positive, and hence the Bogomol'nyi bound is exceeded and there is no supersymmetry, unless either Q_1 or Q_2 vanishes.

For $N = 3$, we find the following polynomial solution of the $SL(4, R)$ Toda equations:

$$\begin{aligned} e^{-q_1} &= a_0 + a_1 \rho + a_2 \rho^2 + \frac{1}{6} \rho^3, \\ e^{-q_2} &= a_1^2 - 2a_0 a_2 + (2a_1 a_2 - a_0) \rho + 2a_2^2 \rho^2 + \frac{2}{3} a_2 \rho^3 + \frac{1}{12} \rho^4, \\ e^{-q_3} &= a_0 - 4a_1 a_2 + 8a_2^3 + (4a_2^2 - a_1) \rho + a_2 \rho^2 + \frac{1}{6} \rho^3, \end{aligned} \quad (2.28)$$

where the constants a_0 , a_1 and a_2 are determined in terms of the charges Q_1 , Q_2 and Q_3 by the requirement that the dilatonic scalars vanish at $\rho = 0$. This implies that

$$e^{q_\alpha(0)} = \prod_{\beta} (4Q_\beta)^{4(M^{-1})_{\alpha\beta}}, \quad (2.29)$$

and hence

$$\begin{aligned} a_0 &= \frac{1}{64} Q_1^{-3/2} Q_2^{-1} Q_3^{-1/2}, \quad a_1^2 - 2a_0 a_2 = \frac{1}{256} Q_1^{-1} Q_2^{-2} Q_3^{-1}, \\ a_0 - 4a_1 a_2 + 8a_2^3 &= \frac{1}{64} Q_1^{-1/2} Q_2^{-1} Q_3^{-3/2}. \end{aligned} \quad (2.30)$$

The metric is given by (2.13), with

$$e^{2B} = \prod_{\alpha} e^{q_\alpha(0) - q_\alpha}, \quad (2.31)$$

and hence

$$m = \frac{a_1}{4a_0} + \frac{2a_1 a_2 - a_0}{4(a_1^2 - 2a_0 a_2)} + \frac{4a_2^2 - a_1}{4(a_0 - 4a_1 a_2 + 8a_2^3)}. \quad (2.32)$$

Thus we find that the mass is given in terms of the charges by the positive root of the sextic

$$\begin{aligned} &m^6 - (3Q_1^2 + 2Q_1 Q_3 + 3Q_3^2 + 3Q_2^2) m^4 - 36\sqrt{Q_1 Q_3} Q_2 (Q_1 + Q_3) m^3 \\ &+ \left[(Q_1 + Q_3)^2 (3Q_1^2 - 2Q_1 Q_3 + 3Q_3^2) - Q_2^2 (21Q_1^2 + 122Q_1 Q_3 + 21Q_3^2) + 3Q_2^4 \right] m^2 \\ &+ 4\sqrt{Q_1 Q_3} Q_2 (Q_1 + Q_3) (9Q_1^2 - 14Q_1 Q_3 + 9Q_3^2 - 18Q_2^2) m \\ &- (Q_1 - Q_3)^2 (Q_1 + Q_3)^4 - Q_2^2 (3Q_1^4 - 68Q_1^3 Q_3 + 114Q_1^2 Q_3^2 - 68Q_1 Q_3^3 + 3Q_3^4) \\ &- Q_2^4 (3Q_1^2 + 38Q_1 Q_3 + 3Q_3^2) - Q_2^6 = 0. \end{aligned} \quad (2.33)$$

There seems to be no way to give an explicit closed-form expression for the mass in terms of the charges. The Bogomol'nyi matrix $\mathcal{M} = m\mathbb{1} + Q_1\Gamma_{\hat{1}\hat{2}12} + Q_2\Gamma_{\hat{1}\hat{2}23} + Q_3\Gamma_{\hat{1}\hat{2}34}$ has eigenvalues

$$\mu = m \pm \sqrt{(Q_1 \pm Q_3)^2 + Q_2^2} , \quad (2.34)$$

where the two \pm signs are independent. For generic values of the charges, $\mu > 0$ and the solution has no supersymmetry. If $Q_2 = 0$, the solution reduces to the two-charge supersymmetric solution, preserving $\frac{1}{4}$ of the supersymmetry. In this case, the $SL(4, R)$ Toda equations reduce to two decoupled Liouville equations.

For higher values of N , the explicit forms of the polynomial solutions to the $SL(N+1, R)$ Toda equations become increasingly complicated. The structure of these polynomials can be summarised as follows. For each N , we find that e^{-q_α} are polynomials in ρ of degree $n_\alpha = \alpha(N+1-\alpha)$, *i.e.*

$$\frac{d^{n_\alpha+1}}{d\rho^{n_\alpha+1}} e^{-q_\alpha} = 0 . \quad (2.35)$$

After substituting these into the $SL(N+1, R)$ Toda equations (2.17), we find that there are N independent parameters, which can be related to the N charges Q_α by equation (2.29). The metric is given by (2.13) with e^{2B} again given by (2.31). The mass is given in terms of charges by an $N!$ -th-order polynomial equation. Although it appears not to be possible to give closed-form expressions for the mass in terms of the charges for $N \geq 3$, we expect nevertheless that it is less than the sum of the charges, indicating again that they are bound states with negative binding energies. One can see this explicitly in the special case where the charges have the fixed ratio given by

$$Q_\alpha = aQ \left(\sum_{\beta} (M^{-1})_{\alpha\beta} \right)^{1/2} = \frac{1}{2}aQ \sqrt{\alpha(N+1-\alpha)} , \quad (2.36)$$

where a is given by (1.7). Under these circumstances the solutions reduce to single-scalar solutions, given by (2.3), and have mass

$$m = \frac{2Q}{a} . \quad (2.37)$$

It is easy to verify that this is always larger than the total mass of the widely-separated constituents, $m_\infty = \sum_\alpha Q_\alpha$. The calculation of the eigenvalues of the Bogomol'nyi matrix becomes increasingly complicated with increasing N . For example, for the $SL(5, R)$ case we find

$$\mu = m \pm \sqrt{Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 \pm 2\sqrt{(Q_1Q_3)^2 + (Q_1Q_4)^2 + (Q_2Q_4)^2}} , \quad (2.38)$$

whilst for $SL(6, R)$ we find that $\mu = m \pm \kappa$, where κ denotes the roots of the quartic equation $\kappa^4 - 2\kappa^2 \alpha - 8\kappa Q_1 Q_3 Q_5 + \beta = 0$, and

$$\begin{aligned}\alpha &= Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 + Q_5^2, \\ \beta &= a^2 - 4\left((Q_1 Q_3)^2 + (Q_1 Q_4)^2 + (Q_1 Q_5)^2 + (Q_2 Q_4)^2 + (Q_2 Q_5)^2 + (Q_3 Q_5)^2\right).\end{aligned}\tag{2.39}$$

For all N , the solutions are non-supersymmetric for generic values of the charges. However, they can be reduced to the previously-known supersymmetric solutions if appropriate charges are set to zero, such that the remaining charges Q_α have non-adjacent indices. In these cases, the solutions preserve 2^{-n} of the supersymmetry, where n is the number of charges remaining.

3 Toda Instantons

In the previous section we discussed $(D - 3)$ -branes solutions, where the 1-form field strengths carry magnetic charges. We can also discuss the case where the 1-forms instead carry electric charges. In this case, the solutions will describe p -branes with $p = -1$, which can be interpreted as instantons. Since there is no longer a time direction, it is necessary first to Euclideanise the supergravity theories, by performing a Wick rotation of the time coordinate. At the same time, account must be taken of the parities of the various fields in the theory, since the kinetic terms for fields of odd parity will undergo a sign change. This has been discussed in detail for the Type IIB superstring in [20], where it was argued that the kinetic term for the R-R scalar χ undergoes such a reversal of sign. In our case, where we keep the subset of 1-forms $\mathcal{F}_1^{(12)}, \mathcal{F}_1^{(23)}, \mathcal{F}_1^{(34)}, \dots$ in $D \leq 9$ dimensions, the potential for $\mathcal{F}_1^{(12)}$ is precisely this same field χ if we follow the type IIB reduction route. By the same token, we may argue that the potentials for the other 1-forms should also be viewed as having odd parity. As a check, one may verify that this assignment is consistent with the structure of the Chern-Simons modifications for the entire set of field strengths in the supergravity theory. Thus we are led to replace the consistently-truncated Lagrangian (2.12) by

$$e^{-1}\mathcal{L} = R - \frac{1}{2} \sum_{\alpha, \beta=1}^N (M^{-1})_{\alpha\beta} \partial_M \varphi_\alpha \partial^M \varphi_\beta + \frac{1}{2} \sum_{\alpha=1}^N e^{-\varphi_\alpha} (\partial \chi_\alpha)^2 \tag{3.1}$$

in the Euclideanised theory.

In the instanton solutions, the metric is flat, and can be written as

$$ds^2 = dr^2 + r^2 d\Omega^2, \tag{3.2}$$

where $d\Omega^2$ is the metric on the unit $(D-1)$ -sphere. The standard elementary ansatz for the 1-form field strengths becomes simply

$$F_\alpha = d\chi_\alpha = 4Q_\alpha e^{\varphi_\alpha} dr , \quad (3.3)$$

and the remaining equations of motion become

$$\varphi''_\alpha = \frac{8}{(D-2)^2} \sum_\beta Q_\beta^2 e^{\varphi_\beta} , \quad (3.4)$$

where a prime denotes the derivative with respect to $\rho \equiv r^{2-D}$. Following analogous steps to those described in the previous section, we find that (3.4) reduces to the $SL(N+1, R)$ Toda equations (2.17), where

$$\varphi_\alpha = \frac{1}{2} \sum_\beta M_{\alpha\beta} q_\beta - 2 \log \left(\frac{4Q_\alpha}{D-2} \right) . \quad (3.5)$$

We now obtain the desired instanton solutions by taking the same solutions of the Toda equations that we discussed in the previous section, namely those that are finite polynomials in ρ . It is worth remarking, however, that since ρ is now equal to r^{2-D} rather than to $\log r$, the instanton solutions are well-behaved asymptotically at large r .

4 $SL(3, R)$ Toda p -branes in the type IIB string

Having obtained the non-supersymmetric $SL(N+1, R)$ Toda $(D-3)$ -branes and instanton solutions in $3 \leq D \leq 8$ dimensions, it is of interest to investigate how these solutions be interpreted in M-theory in $D = 11$ or string theory in $D = 10$. All these Toda p -brane solutions involve 1-form field strengths $\mathcal{F}^{(\alpha, \alpha+1)}$ which arise from the dimensional reduction of the metric in $D = 11$. Thus the corresponding 11-dimensional oxidations of these solutions involve a twisted metric. On the other hand, from the type IIB string point of view, the field strength $\mathcal{F}^{(12)}$ is the dimensional reduction of the derivative of the R-R scalar χ , and $\mathcal{F}^{(23)}$ is the dimensional reduction of the NS-NS 3-form field strength. Thus the Toda solutions that involve only $\mathcal{F}^{(12)}$ and $\mathcal{F}^{(23)}$ can be oxidised into type IIB solutions in $D = 10$ whose metrics will be diagonal. For example the two-charge 5-brane in $D = 8$ can be oxidised to a 5-brane intersecting a 7-brane in the $D = 10$ type IIB theory. The metric is given by

$$ds^2 = T_2^{-1/4} dx^\mu dx^\nu \eta_{\mu\nu} + T_2^{3/4} T_1 (dr^2 + r^2 d\theta^2) + T_2^{3/4} (dz_1^2 + dz_2^2) , \quad (4.1)$$

where T_1 and T_2 are given by (2.25) with $\rho = \log r$. This describes a 7-brane if $Q_2 = 0$; if instead $Q_1 = 0$, it describes a planar continuum of 5-branes. The interpolation when both

Q_1 and Q_2 are non-zero provides the interpretation as an intersection of a 5-brane and a 7-brane.

On the other hand, the two-charge instanton solution in $D = 8$ can be oxidised to a string intersecting an instanton in $D = 10$ type IIB theory with Euclidean signature. The metric is

$$ds^2 = T_2^{-3/4}(dz_1^2 + dz_2^2) + T_2^{1/4}(dr^2 + r^2 d\Omega^2) , \quad (4.2)$$

and $e^\phi = T_1 T_2^{-1/2}$, where T_1 and T_2 are given by (2.25) with $\rho = r^{-6}$. When $Q_1 = 0$, it is nothing but the NS-NS string of type IIB theory in $D = 10$ Euclidean space. If instead $Q_2 = 0$, the space is flat because of the cancellation of the energy-momentum tensors between the dilaton and the R-R scalar. It describes a planar continuum of supersymmetric instantons, of the type obtained in type IIB supergravity in [20].

It is worth remarking that the NS-NS string solution in (4.2) carries a real charge, if the NS-NS 3-form has odd parity. In fact U duality of the type IIB theory in Euclidean space requires that if the R-R scalar has odd parity, then one of the 3-forms must have odd parity and the other must have even parity. Thus U duality interchanges real and imaginary 3-form charges in the Euclidean type IIB string, which is reminiscent of the electromagnetic duality in the Euclidean $D = 4$ Maxwell equations [21]. With the opposite choice of parity assignments for the 3-forms, the string in (4.2) would carry imaginary electric charge, which would be analogous to the Euclidean black holes with imaginary electric charge discussed in [21]. It is not altogether clear in the supergravity case, however, what the proper parity assignments should be. For example, the $\mathcal{F}_1^{(12)}$ field strength was argued to have odd parity in the type IIB theory [20]. However, this same field strength, reduced to $D = 9$, can also be obtained from the dimensional reduction of the metric in $D = 11$ Euclidean supergravity, which seems to suggest that it should instead have even parity. Possibly any assignment of parities that is compatible with U duality is permissible.

5 Conclusions and discussion

The bosonic Lagrangian for maximal supergravity dimensionally reduced to D dimensions is in general rather complicated, owing to the occurrence of Chern-Simons modifications to many of the field strengths. In certain rather special circumstances, p -brane solutions can be found that correspond to consistent truncations of the Lagrangian in which only field strengths without Chern-Simons modifications are present. Solutions using such subsets of the fields are much simpler in form than the more generic ones, and therefore are also much

easier to obtain. Most of the known solutions of this simpler form are supersymmetric. The equations of motion that describe them can be recast in the form of one or more decoupled Liouville equations. A non-supersymmetric example was also already known, namely an $a = \sqrt{3}$ black-hole dyon in $D = 4$ [13, 14]. The equations of motion in this case can be cast into the form of the $SL(3, R)$ Toda equations [6]. In this paper, we have investigated a further class of non-supersymmetric solutions, in which certain of the 1-form field strengths carry magnetic or electric charges, giving rise to $(D - 3)$ -branes or instantons respectively. The equations of motion again turn out to be those of completely integrable systems, namely the $SL(N + 1, R)$ Toda equations. These solutions reduce to the previously-known multiply-charged supersymmetric solutions when appropriate charges vanish.

In most of the paper, we concentrated on describing the D -dimensional $SL(N + 1, R)$ Toda p -branes in terms of the Kaluza-Klein reduction of M-theory. We showed also that there is an $SL(3, R)$ Toda solution in the type IIB theory in $D = 10$, which is obtained by oxidising the $SL(3, R)$ Toda 5-brane in $D = 8$ *via* the alternative type IIB pathway. The $D = 10$ solution has the interpretation of an intersection between a 7-brane and a 5-brane. In particular, when the charge associated with the 5-brane vanishes, it gives rise to a supersymmetric 7-brane in the type IIB theory. In this case, the function T_2 in (4.1) becomes harmonic. If it is now taken to depend on only one of the two transverse-space Cartesian coordinates $y_1 = r \cos \theta$ or $y_2 = r \sin \theta$, then one obtains the 7-brane solution discussed in [22], where T_2 is a linear function of the chosen coordinate. On the other hand χ is proportional to the other transverse coordinate, which must therefore be periodic because of the $\chi \rightarrow \chi + 1$ shift symmetry of the R-R scalar. (Compactifying the type IIB theory on this periodic coordinate gives rise to massive $N = 2$ supergravity in $D = 9$ [22].) Both of the above 7-branes in the type IIB theory should be distinguished from the modular-invariant 7-brane [20], which is the oxidation of the cosmic string in $D = 4$ [23].

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